

PEAK POINTS FOR PSEUDOCONVEX DOMAINS: A SURVEY

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1. Introduction

Let D be a domain in \mathbf{C}^n with smooth (that is, C^∞) boundary. If $0 \leq \alpha \leq \infty$ we denote by $A^\alpha(D)$ the space of functions holomorphic on D and of class C^α on \overline{D} . We write $A(D)$ for $A^0(D)$ and $A^\omega(D)$ for the space of functions holomorphic on a neighborhood of \overline{D} . We say that a point $p \in \partial D$ is a peak point relative to D for $A^\alpha(D)$ if there exists a function $f \in A^\alpha(D)$ so that $f(p) = 1$ and $|f| < 1$ on $\overline{D} \setminus \{p\}$. We call f a peak function. This condition is clearly equivalent to the existence of a strong support function, that is, a function $g \in A^\alpha(D)$ so that $g(p) = 0$ and $\operatorname{Re} g > 0$ on $\overline{D} \setminus \{p\}$. We say that $p \in \partial D$ is a local peak point for $A^\alpha(D)$ if p is a peak point for $A^\alpha(D \cap V)$ for some neighborhood V of p .

We want to determine whether boundary points are peak points. Finding a peak function can be thought of as giving a quantitative converse to the maximum modulus principle. Now observe that if f is a peak function at p relative to D then $1/(1 - f)$ is a holomorphic function on D with no holomorphic extension past p . Thus if every boundary point of D is a peak point then D is a domain of holomorphy. In light of this fact, if we want to determine whether boundary points are peak points it makes sense to restrict our attention to domains of holomorphy. By the solution of the Levi problem these are the (Levi) pseudoconvex domains. Here is another observation: If there is a complex disc in the boundary of D then (by the maximum modulus principle) no point in the relative interior of that disc can be a peak point for $A(D)$. A natural condition in this context is that D be of finite type at the point p in question, in the sense of D'Angelo: There is a finite upper bound on the order of contact of nontrivial complex analytic varieties with the boundary at p . The infimum of all such upper bounds is called the type at p . We formulate the major open question in terms of this notion.

Question. *If D is a smooth bounded pseudoconvex domain in \mathbf{C}^n of finite type at $p \in \partial D$, is p a peak point for $A^\alpha(D)$ for some $\alpha > 0$?*

In this paper we survey the current knowledge with regard to this question. The question has been answered in the affirmative in \mathbf{C}^2 , but even here there are remaining problems, particularly in connection with the relationship between the regularity exponent α and the type at the boundary point in question. In our exposition we present unpublished results of Laszlo that apply to certain domains in \mathbf{C}^2 . His results give very good regularity and also construct the peak function in a more explicit way than was previously done. The question stated above is still open in \mathbf{C}^n for $n > 2$. In fact, it is not known whether every boundary point of a smooth bounded pseudoconvex domain D of finite type in \mathbf{C}^n is a peak point for $A(D)$.

In Section 2 we provide some background information and analyze the case of strict pseudoconvexity. Section 3 discusses the notion of finite type, presents the Kohn-Nirenberg domain for which no smooth peak function exists at a point, and considers notions of strict type. In Section 4 we first outline the solution in \mathbf{C}^2 by Bedford and Fornæss of the main question above. Then we present the modification by Laszlo of the Bedford-Fornæss construction. Section 5 discusses other approaches to the problem, and Section 6 presents what is known about the question in \mathbf{C}^n . The concluding section is a miscellany of results.

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2. Background information and the strictly pseudoconvex case

Let D be a domain in \mathbf{C}^n with smooth boundary, and let r be a smooth defining function for D , in the following sense: For some neighborhood V of ∂D , the real-valued function r is smooth on V ,

$$V \cap D = \{z \in V : r(z) < 0\},$$

and $\nabla r \neq 0$ on V . For $p \in \partial D$ and $t = (t_1, \dots, t_n) \in \mathbf{C}^n$ we write $\partial r_p(t)$ for

$$\sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j,$$

and we write $L_p(r, t)$ for

$$\sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k ,$$

the Levi form of r at p applied to t . The domain D is said to be (Levi) pseudoconvex if for each $p \in \partial D$ we have $L_p(r, t) \geq 0$ whenever $\partial r_p(t) = 0$. The geometric interpretation of the condition $\partial r_p(t) = 0$ is that t belongs to the complex tangent space to the boundary at p , that is, the maximal complex subspace of the real tangent space at p . The domain is said to be strictly pseudoconvex at a boundary point p if $L_p(r, t) > 0$ whenever $\partial r_p(t) = 0$ and $t \neq 0$. Note that the complex tangent space to the boundary is trivial for domains in \mathbf{C} with smooth boundary, so such domains are strictly pseudoconvex at each boundary point. In general the condition of strict pseudoconvexity is independent of the choice of defining function, and it implies that there exist a defining function ρ and a constant c so that

$$(*) \quad L_p(\rho, t) \geq c \|t\|^2, t \in \mathbf{C}^n.$$

In fact, if r is a defining function the choice $\rho = r + Mr^2$ satisfies condition $(*)$ for M sufficiently large and c sufficiently small.

This observation makes it easy to construct a local peak function at a strictly pseudoconvex boundary point. The following result in some form is usually attributed Levi, who introduced the notion of pseudoconvexity we have presented (see [45] and [46]).

2.1 Proposition. *If D is a smooth bounded domain in \mathbf{C}^n that is strictly pseudoconvex at $p \in \partial D$ then p is a local peak point for $A^\omega(D)$.*

Proof: For ease of notation we assume that 0 is a strictly pseudoconvex boundary point of D . Choose a defining function ρ for which condition $(*)$ holds. For $w = (w_1, \dots, w_n) \in \mathbf{C}^n$ put

$$g(w) = -2\partial\rho_0(w) - \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k ,$$

a holomorphic polynomial vanishing at 0. Now the complex form of the Taylor expansion of ρ about 0 gives

$$\rho(z) = -\operatorname{Re} g(z) + L_0(\rho, z) + o(\|z\|^2).$$

But if $z \in \overline{D}$ then $\rho(z) \leq 0$ and we have

$$\operatorname{Re} g(z) \geq L_0(\rho, z) + o(\|z\|^2).$$

Hence, by condition $(*)$, g is a local strong support function at 0. \square

Note that the local strong support function is expressed in an elementary way in terms of the second-order Taylor expansion of the special defining function.

To get a global peak function we need the following result due to Kohn (see [39] and pages 229–231 of [40]).

2.2 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbb{C}^n and ϕ a smooth $(0,1)$ -form on \overline{D} , in the sense that $\phi = \phi_1 d\bar{z}_1 + \cdots + \phi_n d\bar{z}_n$ with $\phi_j \in C^\infty(\overline{D})$ for all j . Assume that ϕ is $\bar{\partial}$ -closed, that is, $\frac{\partial \phi_j}{\partial \bar{z}_k} = \frac{\partial \phi_k}{\partial \bar{z}_j}$ for all j, k . Then there exists $\psi \in C^\infty(\overline{D})$ so that $\bar{\partial}\psi = \phi$, in the sense that $\frac{\partial \psi}{\partial \bar{z}_j} = \phi_j$ for all j .*

Now we can show that every strictly pseudoconvex boundary point of a smooth bounded pseudoconvex domain is a (global) peak point. This was proved by Hakim and Sibony (see [32], [33]) and by Pflug in [53].

2.3 Theorem. *If D is a smooth bounded pseudoconvex domain in \mathbb{C}^n that is strictly pseudoconvex at $p \in \partial D$ then p is a peak point for $A^\infty(D)$.*

Proof: Apply Proposition 2.1 to get a local smooth strong support function g at p . Choose a nonnegative C^∞ function χ with support in a small neighborhood of p so that $\chi = 1$ near p , and define the $(0,1)$ -form ϕ on \overline{D} by $\phi(p) = 0$ and $\phi = \bar{\partial}(\chi/g)$ away from p . Then ϕ is smooth and $\bar{\partial}$ -closed, so by Theorem 2.2 there exists $\psi \in C^\infty(\overline{D})$ so that $\bar{\partial}\psi = \phi$. Adjust ψ by adding a constant to ensure that it has negative real part. Now define G on \overline{D} by $G(p) = 0$ and $G = 1/(\chi/g - \psi)$ away from p . It is easy to see that G belongs to $A^\infty(D)$ and is a strong support function at p relative to D . \square

2.4 Remark.

1) The method of argument in the proof of Theorem 2.3 can be adapted as follows. Let p be a boundary point of a smooth bounded pseudoconvex domain D . Suppose we are given a local peak function f at p belonging to $A^\alpha(D)$ for some $\alpha \leq \infty$. Assume that f is smooth up to the boundary away from p . Then p is a (global) peak point for $A^\alpha(D)$. Because of this fact, in the subsequent exposition we will focus on constructing such local peak functions. Typically the local peak function

will extend to be holomorphic across the boundary away from p . Sometimes the domain under consideration will be unbounded; Theorem 2.2 has been extended by Gay and Sebbar to unbounded pseudoconvex domains with smooth boundary. See Théorème 4.1 of [30].

2) In Theorem 4.4 of [58], Rossi proved that if \overline{D} has a neighborhood basis of pseudoconvex domains then every local peak point for $A^\omega(D)$ is a peak point for $A^\omega(D)$. If D strictly pseudoconvex at every boundary point then it is easy to see that \overline{D} has a neighborhood basis of pseudoconvex domains, so every boundary point is a peak point for $A^\omega(D)$. This condition on the neighborhood basis also holds for pseudoconvex domains of finite type, but the proof is much more difficult. Rossi's result is not valid without this condition on the neighborhood basis: In [15], Diederich and Fornæss defined a family of smooth bounded pseudoconvex domains with no such neighborhood basis. (These “worm” domains have a complex annulus in the boundary.) In Theorem 3 of that paper they showed that there are strictly pseudoconvex boundary points that are not (global) peak points for A^ω .

3) In [59], Sibony constructed a smooth bounded pseudoconvex domain D in \mathbf{C}^3 having the following property: There is a point $p \in \partial D$ that is a local peak point for $A(D)$ but is not a peak point. Relying on the method of proof we described in Theorem 2.3, Sibony draws the following conclusion: There exists a $(0, 1)$ -form ϕ on D whose coefficients are smooth on D and continuous on \overline{D} so that ϕ is $\bar{\partial}$ -closed but no solution of $\bar{\partial}\psi = \phi$ is bounded on D . (Compare this with Theorem 2.2.)

4) Let D be a smooth bounded pseudoconvex domain. Assume that D is of finite type (as defined in Section 1), or more generally that ∂D contains no nontrivial complex manifold. It turns out that the set of boundary points at which D is strictly pseudoconvex is dense in the boundary of D . (If not, by Freeman's work in [28] there would be a local foliation of the boundary by complex submanifolds, a contradiction. For a precise statement in the case of finite type see the Main Theorem of Catlin's paper [9].) By Theorem 2.3, then, for pseudoconvex domains of finite type the set of peak points for $A^\infty(D)$ is dense in the boundary of D . Another result along these lines is due to Basener: In [1] he showed that if p is a C^2 boundary point of an open set D in \mathbf{C}^n and p is a peak point for $A(D)$ then p is a limit of strictly pseudoconvex boundary points of D .

The following standard result is basic for studying peak functions with some regularity. For a proof see, e.g., Proposition 12.2 of [27].

2.5 Hopf lemma. *Let Ω be a bounded domain in the complex plane with C^2 boundary and u a negative subharmonic function on Ω . For $z \in \Omega$ denote by $\delta(z)$ the distance from z to $\partial\Omega$. Then there exists a constant $c > 0$ so that $u(z) \leq -c\delta(z)$ for all $z \in \Omega$.*

Now we apply the Hopf lemma to peak functions.

2.6 Holomorphic Hopf lemma. *Let D be an open set in \mathbf{C}^n with C^2 boundary near $p \in \partial D$. Assume that p is a local peak point for $A^1(D)$ with strong support function g . Then the derivative at p of $\operatorname{Re} g$ in the direction of the outward normal is negative.*

Proof: Apply the Hopf lemma (2.5) to $-\operatorname{Re} g$ near p on the intersection of D with the complex line spanned by a normal to ∂D at p . \square

We conclude this section with one version of the Bishop 1/4–3/4 method for constructing peak functions.

2.7 Theorem. *Let D be a bounded domain in \mathbf{C}^n and $p \in \partial D$. Assume that for every neighborhood V of p there exists $f \in A(D)$ so that $|f| \leq 1$ on \overline{D} , $f(p) > 3/4$, and $|f| < 1/4$ on $\overline{D} \setminus V$. Then p is a peak point for $A(D)$.*

The method in fact applies much more generally to construct a peak function from a family of approximate peak functions. The idea of the proof is to construct inductively a nested sequence of neighborhoods of p and to add (scaled) approximate peak functions for those neighborhoods. For a statement and proof in the case of certain closed subspaces of $C(X)$ (with X compact Hausdorff), see Theorem 2.3.2 of [8].

3. Finite type, the Kohn-Nirenberg domain, and strict type

Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n . Recall that D is of finite type at a point $p \in \partial D$ if there is a finite upper bound on the order of contact of complex analytic varieties with the boundary at p . The infimum of all such upper bounds is the type at p . We remark that every bounded pseudoconvex domain with real-analytic boundary is of finite type. See D'Angelo's book [14] for further background. In this section we will re-formulate this notion for domains in \mathbf{C}^2 . In addition, we consider certain model domains having no smooth peak

function at a point. We also consider so-called strict type conditions in \mathbf{C}^n .

First we make the following observation. Fix a domain D in \mathbf{C}^n with smooth boundary and a point $p \in \partial D$. Suppose we are given local holomorphic coordinates (z, w) , with $z \in \mathbf{C}^{n-1}$ and $w = u + iv$, in which $p = 0$ and u points in the outward normal direction to ∂D at p . Then D has a smooth defining function of the form

$$u + A(z) + B(z)v + O(|v|^2),$$

where A vanishes to order at least 2 at 0 and $B(0) = 0$. Note that A and B depend on the choice of coordinates. In the rest of this paper we refer to this expression as a standard form for a defining function in these coordinates.

Now we restrict attention to pseudoconvex domains in \mathbf{C}^2 and use the preceding setup to re-formulate the notion of finite type. If in standard form for some coordinates we have $A \equiv 0$, then the complex manifold $w = 0$ lies in the boundary, so D is not of finite type at p . Now assume that $A \not\equiv 0$ and write

$$A(z) = P(z) + O(|z|^{m+1})$$

with $P \not\equiv 0$ a polynomial (in $z \in \mathbf{C}$ and \bar{z}) that is homogeneous of degree $m \geq 2$. Note that P depends on the choice of coordinates. Then D is of finite type at p if and only if there exist such coordinates in which P is not harmonic. Further, the degree m of such a P is uniquely determined (independent of the choice of coordinates and defining function) and is the type at p . For an explanation of this re-formulation, see Lecture 28 of [27]. It is easy to see that the pseudoconvexity of D implies that P is subharmonic. In particular, the type is an even integer.

The case of type 2 corresponds to strict pseudoconvexity, and each such point is a local peak point for $A^\omega(D)$ and a global peak point for $A^\infty(D)$ by the results of the preceding section. It turns out that in \mathbf{C}^2 points of type 4 have the same property. We sketch the proof to indicate how similar results are proved, but we leave details to be filled in by the indicated references.

3.1 Proposition. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^2 . If D is of type 4 at $p \in \partial D$ then p is a local peak point for $A^\omega(D)$.*

Proof: We use the preceding notation. Choose coordinates so that P

is a subharmonic polynomial homogeneous of degree 4. After a holomorphic change of coordinates we may assume that P has the form

$$P(z) = |z|^4 + t|z|^2 \operatorname{Re}(z^2)$$

with $t \geq 0$. Because P is subharmonic we have $t \leq 4/3$. Now the key point is that if $\epsilon > 0$ is sufficiently small then there exists $c > 0$ so that

$$|z|^4 + t|z|^2 \operatorname{Re}(z^2) + \frac{t}{4(1-\epsilon)} \operatorname{Re}(z^4) \geq c|z|^4.$$

(See Lecture 28 of [27].) Now replace w by $w + \frac{t}{4(1-\epsilon)} z^4$ for small $\epsilon > 0$. In the new coordinates the corresponding homogeneous polynomial P is the expression on the left-hand side of the preceding inequality. Now we analyze the function B . If the leading term in the expansion of B about 0 is linear, say $\operatorname{Im}(\beta z)$ for some $\beta \in \mathbf{C}$, replacing w by $w(1 + \beta z)$ results in a new expansion in which B vanishes to order at least 2 at 0. Further, the polynomial P is not changed. (See Lemma 2.8 and the subsequent discussion in [6] or the proof of Proposition 1.1 in [26].) Hence there exists a constant $T > 0$ so that

$$B^2(z) < TA(z)$$

if $z \neq 0$ is sufficiently small. It follows that $-w - Mw^2$ is a local strong support function if M is a large constant. (See Lemma 2.2 of [6].) \square

The last part of this proof is a special case of a general result due to Bloom. To explain this we use the standard form involving the functions A and B developed for general domains.

3.2 Proposition. *Let D be a domain in \mathbf{C}^n with smooth boundary and $p \in \partial D$. Then p is a local peak point for $A^\omega(D)$ if and only if the following holds: There exist local holomorphic coordinates in which, for some $T > 0$, $B^2(z) < TA(z)$ if $z \neq 0$ is sufficiently small.*

For the proof we refer the reader to Lemma 2.2 and Lemma 2.4 of [6]. We remark that in Bloom's characterization the local peak function is given explicitly in terms of the special defining function in local coordinates: The local strong support function he constructs has the form $-w - Mw^2$ for some constant M , as in the proof of our Proposition 3.1.

We have seen that, for smooth bounded pseudoconvex domains in \mathbb{C}^2 , points of type 2 and points of type 4 are local peak points for A^ω . For points of type 6 there is no such result. The following is due to Fornæss [21].

3.3 Proposition. *Fix a number $t > 0$. Consider the domain D defined near the origin in \mathbb{C}^2 by*

$$\operatorname{Re} w + |z|^6 + t|z|^2 \operatorname{Re}(z^4) + |zw|^2 < 0.$$

If $t \leq 9/5$ then D is pseudoconvex (and strictly pseudoconvex away from the origin), and if $t > 1$ the origin is not a local peak point for $A^1(D)$.

Proof of weaker result: We assume that $1 < t \leq 9/5$. It is straightforward to verify the pseudoconvexity claims, and we show only that there is no local strong support function at the origin in A^ω . (See the comments following the proof.) Assume for a contradiction that such a function g exists. Note that if $(z, w) \in \overline{D}$ then $(iz, w) \in \overline{D}$. Hence by averaging we may assume that $g(iz, w) \equiv g(z, w)$. Now by the holomorphic Hopf lemma (2.6) and the implicit function theorem, near the origin we can write the zero set of g as a graph $w = h(z)$ for some function h holomorphic near 0 satisfying $h(0) = 0$. For ease of notation put

$$P(z) = |z|^6 + t|z|^2 \operatorname{Re}(z^4).$$

Note that P takes negative values in every neighborhood of 0 because $t > 1$. Now the zero set of g does not intersect D near the origin, and using the defining function for D gives

$$\operatorname{Re} h(z) + P(z) + |zh(z)|^2 \geq 0$$

for z near 0. This shows that $h \not\equiv 0$ because P takes negative values. By the symmetry of g we have $h(iz) \equiv h(z)$, so the leading term of the Taylor expansion of h about 0 is cz^{4k} for some positive integer k and constant c . But the preceding inequality and the fact that P takes negative values shows that $k < 2$, and because $\operatorname{Re}(cz^4)$ takes negative values we have a contradiction. \square

A careful examination of the preceding proof will show that the origin is not a local peak point for A^6 . (See also page 402 of [33].) To prove that there is no peak function in A^1 , Fornæss in [21] starts with the Hopf lemma as above and obtains a normal family of harmonic

functions by examining the (normalized) values of a strong support function on slices of the domain by the planes $w = -\epsilon$.

Shortly after Fornæss proved this result, in [2] Bedford and he constructed, at each boundary point of a pseudoconvex domain of finite type in \mathbf{C}^2 , a peak function in A^α for some $\alpha > 0$. In the next section we will discuss this result and also see from Laszlo's work that for the domain in Proposition 3.3 (in the case $t = 9/5$) we may take $\alpha = 1/18$. It is an open problem to determine for which values of $\alpha < 1$ there is a peak function at the origin in A^α relative to this domain. The current methods seem incapable of producing a peak function in A^α for α greater than $1/6$, the reciprocal of the type. In particular, it is not known whether the origin is a peak point for A^α with $\alpha < 1$ arbitrarily close to 1.

It is clear that certain properties of the homogeneous polynomial

$$P(z) = |z|^6 + t|z|^2 \operatorname{Re}(z^4)$$

appearing in the defining function are crucial for the proof of Proposition 3.3. This polynomial is a special case (with $k = 3$ and $\ell = 2$) of a more general two-term homogeneous subharmonic polynomial, namely

$$P(z) = |z|^{2k} + t|z|^{2k-2\ell} \operatorname{Re}(z^{2\ell}),$$

where k and ℓ are positive integers so that $\ell < k$, and $0 < t \leq k^2/(k^2 - \ell^2)$. Using such a polynomial to define a domain as in Proposition 3.3 gives a pseudoconvex domain of type $2k$ at the origin. The seminal study of such a domain was the paper [41] by Kohn and Nirenberg, which considered the case $k = 4$ and $\ell = 3$ with $t = 15/7$. They showed that if the zero set of a function holomorphic near the origin passes through the origin then the zero set intersects the domain. (See also Proposition 1.2 of [60].) This was quite an unexpected result; apparently the expectation in the early 1970s was that some sort of convexity condition would carry over from the strictly pseudoconvex case.

We remark that for such a non-convexity result one needs not only that $t > 1$ but also that ℓ does not divide k (consider the case of type 4). In fact, if ℓ divides k then the origin is a local peak point for A^ω . Kolář proves this and several related results in [42].

In light of the Kohn-Nirenberg domain there are two possible approaches to studying the existence of peak functions. One is to try to construct peak functions with minimal regularity on domains of finite

type. This was the approach of Bedford and Fornæss, as mentioned above, and this will be the point of view in much of the remainder of our study. Another approach is to add conditions stronger than finite type that guarantee the existence of more regular peak functions. These are often called strict type conditions, and in the remainder of this section we consider them.

In [38], Kohn introduced a strict type condition for domains in \mathbf{C}^2 and showed that for smooth bounded pseudoconvex domains in \mathbf{C}^2 every point of strict type (in that sense) is a local peak point for A^ω . In [55], [56], and [57], Range gave a condition on domains in \mathbf{C}^n that in \mathbf{C}^2 is weaker than Kohn's condition. In terms of the function A appearing in the standard form for a defining function, Range's condition is that there exist holomorphic coordinates so that, for some constant $c > 0$ and some integer N ,

$$A(z) \geq c|z|^N$$

near $z = 0$. In [33], Hakim and Sibony studied this strict type condition as it relates to peak functions. In [6], Bloom generalized Kohn's condition to \mathbf{C}^n and introduced a third notion (weaker than Kohn's and stronger than Range's) of strict type. That paper gives a very helpful discussion of all three conditions. To avoid technicalities we give only the main ideas.

First we state one main result of Hakim and Sibony from [33]. The proof is omitted.

3.4 Theorem. *Let D be a smooth bounded domain (not necessarily pseudoconvex) in \mathbf{C}^n and p a boundary point of strict type in the sense of Range: There exist holomorphic coordinates near p so that, for some constant $c > 0$ and some integer N ,*

$$A(z) \geq c|z|^N$$

near $z = 0$. Then p is a local peak point for $A^\alpha(D)$ for some $\alpha > 1$. In addition, if we may take $N \leq 4$ then p is a local peak point for A^ω .

As we noted above, the strict type condition introduced by Bloom in [6] is stronger than Range's condition. It, too, involves a positive-definite condition, but one expressed in terms of the leading homogeneous part of A in weighted holomorphic coordinates. Bloom proves the following result.

3.5 Theorem. *Let D be a smooth bounded pseudoconvex domain and p a boundary point of strict type in the sense of Bloom. Then p is a local peak point for A^ω .*

Sketch of proof: The proof uses the result we stated as Proposition 3.2: The goal is to verify the inequality that, for some $T > 0$, we have $B^2(z) < TA(z)$ if $z \neq 0$ is sufficiently small. Using successive holomorphic changes of coordinates allows one to ensure that the leading part of B (that is, the part of lowest weight) is not the real part of a holomorphic function. The pseudoconvexity condition then implies that this part is of weight at least half that of the leading part of A . Using the positive-definite condition gives the desired inequality. \square

In [6], Bloom also provided a striking example. Among other things, this example shows that the notion of strict type given by Range does not in general imply the existence of a local peak function in A^ω .

3.6 Proposition. *Consider the domain D defined near the origin in \mathbb{C}^2 by $r < 0$, where*

$$r(z, w) = u + 100(|z|^{10} + |z|^2 \operatorname{Re}(z^8)) + |z|^6 v + |z|^2 v^2.$$

(Here we use (z, w) for coordinates and write $w = u + iv$.) Then D is pseudoconvex (and strictly pseudoconvex away from the origin), and the origin is not a local peak point for $A^\omega(D)$ —in fact, is not a local peak point for $A^{13}(D)$. The origin is, however, of strict type in the sense of Range, and so is a local peak point for $A^\alpha(D)$ for some $\alpha > 1$.

Sketch of proof: It is straightforward to verify the pseudoconvexity claims. Bloom shows that there are no local coordinates in which the basic inequality $B^2(z) < TA(z)$ (for $z \neq 0$ sufficiently small) holds. By Proposition 3.2 this proves that there is no local peak function in $A^\omega(D)$. For the rest, one checks that $r(z, z^{16}) \geq c|z|^{16}$ for some $c > 0$ (see [6] and Lecture 30 of [27]). This implies that the origin is of strict type in the sense of Range, and the concluding statement of the proposition follows from Theorem 3.4. \square

We remark that for this example it is not known for which α between 1 and 13 the origin is a local peak point for A^α .

When Range introduced his notion of strict type, he observed that this condition holds at all boundary points of bounded convex domains with real-analytic boundary. We remark that clearly every such boundary point is a peak point for A^ω : Because the boundary can contain

no line segment, at each boundary point the tangent plane intersects the closure of the domain only at the point of tangency, so there is an affine strong support function. A main motivation for Range's work was to prove Hölder estimates for the $\bar{\partial}$ -equation, and for that study he needed a smoothly varying family of functions satisfying certain estimates. In more recent work, also motivated by the study of the $\bar{\partial}$ -equation, Diederich and Fornæss studied smooth bounded convex domains of finite type. In [18], they gave an explicit formula for a local strong support function in A^ω at each boundary point of a convex domain of finite type. In fact, they constructed a smoothly varying family of such functions satisfying optimal estimates with regard to the order of contact. See also their work in [19], where they extended these results to cover so-called lineally convex domains of finite type. (These are domains of finite type so that through each boundary point there passes a complex hyperplane that does not intersect the domain.) It follows that if a smooth bounded domain of finite type is convex (or, more generally, lineally convex), then every boundary point is a global peak point for A^ω .

In part 4) of Remark 2.4 we noted that, for smooth bounded pseudoconvex domains of finite type, the set of peak points for $A^\infty(D)$ is dense in the boundary. Now we give a refinement of that statement for bounded pseudoconvex domains with real-analytic boundary in \mathbf{C}^2 . It follows from this refinement that the points where the Kohn-Nirenberg behavior occurs form a relatively small subset of the boundary.

The following theorem is drawn from the work of Noell and Stensønes in [52]. (See Proposition 1.1 and Proposition 1.6 there.)

3.7 Theorem. *Let D be a bounded pseudoconvex domain with real-analytic boundary in \mathbf{C}^2 . Then there exists a set $E \subset \partial D$ that is a finite union of singleton sets and real-analytic curves so that every point in $\partial D \setminus E$ is a peak point for $A^\infty(D)$.*

Instead of giving the proof we explain the main point. We say that a curve in the boundary is complex tangential at a point if its tangent line lies in the complex tangent space to the boundary at that point. The key fact is that the Kohn-Nirenberg phenomenon cannot occur at any point belonging to a curve having both of the following properties: the curve is complex-tangential at every point, and the boundary is of constant finite type along the curve. (The curves in the preceding theorem can be taken to be nowhere complex-tangential.) For example,

the following is a consequence of Noell's work in [50]. (In [50] the conclusion is that the curve is locally a peak set for $A^\infty(D)$, but by examining the proof or using Theorem 4.10 of [48] one sees that each point on the curve is a peak point. The notion of peak set is defined in Section 7 below.)

3.8 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^2 and γ a smooth curve in ∂D that is complex tangential at every point. Assume that D is of constant finite type along γ . Then each point of γ is a peak point for $A^\infty(D)$.*

To conclude this section we note Iordan's work in [36] giving a sufficient condition for every boundary point to be a peak point for A^∞ .

4. The sector method

As we noted earlier, Bedford and Fornæss in [2] obtained a positive answer to our main question for domains in \mathbf{C}^2 , as follows.

4.1 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^2 and $p \in \partial D$ a point of finite type. Then there exists $\alpha > 0$ such that p is a peak point for $A^\alpha(D)$.*

In this section we discuss their method for constructing peak functions. They blow up the boundary point p to obtain a line bundle over a Riemann surface whose points are sectors in a cone naturally associated to D at p . Using an existence theorem they produce (in the abstract) a section of this line bundle, and the existence of a peak function follows. Here we also present Laszlo's modification of the sector method. For certain domains he obtains the desired section concretely, in terms of a holomorphic polynomial naturally associated to the defining function. Because the results in his dissertation [44] have never been published, we give several details of his method (but omit the more technical parts).

We recall the discussion at the beginning of Section 3: If p is a point of finite type in the boundary of a smooth bounded pseudoconvex domain D in \mathbf{C}^2 , then there exist local holomorphic coordinates (z, w) , with $w = u + iv$, in which $p = 0$ and D has a smooth defining function of the form

$$u + P(z) + O(|v|^2 + |zv| + |z|^{2k+1}).$$

Here P is a subharmonic, but not harmonic, polynomial on \mathbf{C} that is homogeneous of degree $2k$, the type of D at p . For the time being we

fix such a polynomial P . The method of construction in [2] produces a peak function at 0 relative to the domain

$$D' = \{(z, w) \in \mathbf{C}^2 : \operatorname{Re} w + P(z) < 0\}$$

by constructing an associated function on the open cone

$$C = \{(z, w) \in \mathbf{C}^2 : \operatorname{Re} (w^{2k}) + P(z) < 0\}.$$

Because of a bumping procedure the associated function is in fact defined on a conical neighborhood of C . (We postpone the description of such a bumping until Section 6.) This means that the peak function relative to D' is actually defined on a neighborhood of $\overline{D} \setminus \{p\}$ near p : By bumping C we in effect absorb the terms of higher order in the special defining function for D . We thus obtain a local peak function at p relative to D . In what follows we will drop all reference to the neighborhoods and work just on D' and C .

Now we describe in more detail the construction in [2] of a peak function at 0 relative to D' . As we stated above, the peak function is defined in terms of an associated function that is holomorphic on C . Here is a key observation: Intersecting C with a complex line through the origin results in a finite collection of sectors in that line. The associated function constructed in [2] is linear on each of these sectors. Given such a function g , one can define

$$h(z, w) = \prod_{j=1}^{2k} g(z, e^{i\pi j/k} w)$$

and then define a function H holomorphic on D' by

$$H(z, w^{2k}) = h(z, w)$$

and $H(0, 0) = 0$. If we take a large enough root of H we obtain a strong support function relative to D' . Because we are concerned with the regularity at 0, we note that the function g , it turns out, satisfies a Hölder condition of order 1, so H is of class $C^{1/(2k)}$. Then the strong support function belongs to A^α for some $\alpha > 0$, but we have no estimate for α without information about how g winds around the origin.

The main problem, then, is to construct the function g that is linear on each sector. The difficulty is that the sectors may vary with the complex line in a complicated way; for example, if we follow a given

initial sector around a loop in complex projective space we might end up at a different sector. A key insight of [2] is that the sectors can be considered as points of a Riemann surface over complex projective space. The function g is then obtained from a section of a line bundle over this Riemann surface, as we noted in our introduction to the sector method. Among other things, this construction requires showing that the sectors vary in a smooth way with the complex line, a point we address below. In the end the existence of the section follows by solving a multiplicative Cousin problem.

This is an ingenious method, and the analysis of the domain in terms of these sectors has proven useful in various contexts (for example, [3], [22], [25], and [52]). One drawback is that the resulting peak function is produced by means of an abstract existence theorem, which yields little insight on how the peak function depends on the defining function. The method also gives little control on the regularity of the peak function. Laszlo's modification in [44] of the Bedford-Fornæss construction addresses both of these concerns: The peak function is defined in terms of the roots of an algebraic equation naturally associated to the homogeneous polynomial P , and there is good control on the regularity of the peak function. His method applies when P has a special form. Before we describe this modified construction we take a closer look at how the sectors vary with the complex line.

In a sequence of lemmas, Bedford and Fornæss show how the subharmonicity of P is reflected in the behavior of the sectors. The key point is that each sector has an angular opening of size at most $\pi/(2k)$, and any two sectors must be separated by an angular opening of size at least $\pi/(2k)$. These facts imply that a sector cannot split as the complex line varies, a crucial observation in showing that the sectors vary smoothly.

Here is the main result proved by Laszlo in [44].

4.2 Theorem. *Fix positive integers k and ℓ so that $k/2 \leq \ell < k$, and put $t = k^2/(k^2 - \ell^2)$. Define*

$$P(z) = |z|^{2k} + t|z|^{2k-2\ell} \operatorname{Re}(z^{2\ell}),$$

a homogeneous subharmonic polynomial. Then 0 is a peak point relative to every smooth bounded pseudoconvex domain in \mathbb{C}^2 having a defining function of the form

$$\operatorname{Re} w + P(z) + O(|w|^2 + |zw| + |z|^{2k+1}).$$

Further, the peak function constructed belongs to A^α with $\alpha = 1/(6k)$. Note that $6k$ is 3 times the type of the domain at 0.

We will see that the peak function is defined in terms of the zeros of a holomorphic polynomial naturally associated to the homogeneous polynomial P .

We sketch the proof of the theorem in two parts: the construction of the Riemann surface, which is more concrete than the original construction, and the definition of the function g that is linear on each sector. The first part is valid for any polynomial P on \mathbf{C} that is subharmonic, but not harmonic, and homogeneous of degree $2k$. Hence we describe the construction for any such polynomial. The second part applies only to the special two-term polynomials defined in the theorem.

Construction of the Riemann surface: We write P in the form

$$P(z) = \sum_{j=0}^{2k} c_j z^j \bar{z}^{2k-j}.$$

Note that $\bar{c}_j = c_{2k-j}$ for $0 \leq j \leq 2k$ because P is real-valued, and $c_k > 0$ because P is subharmonic but not harmonic. We assume that the harmonic terms in P have been removed, so $c_0 = c_{2k} = 0$. Put $a_j = c_{k+j}$ for $0 \leq j \leq k-1$, and for $z \in \mathbf{C}$ define

$$Q_0(z) = \sum_{j=0}^{k-1} a_j z^{2j}.$$

We think of this as the polynomial corresponding to intersecting the cone C with the complex line $w = 0$. More generally, for fixed $\zeta \in \mathbf{C}$ the intersection of C with the complex line $w = \zeta z$ is described by

$$\operatorname{Re}(\zeta^{2k} z^{2k}) + P(z) < 0,$$

and we define

$$Q_\zeta(z) = \zeta^{2k} z^{2k} + \sum_{j=0}^{k-1} a_j z^{2j} = \zeta^{2k} z^{2k} + Q_0(z).$$

This is the holomorphic polynomial whose zeros will be used to define the locally linear function g —the function from which the peak function is derived.

The connection between $Q_\zeta(z)$ and the defining function can be explained geometrically as follows. It is easy to see that, if we write $z = |z|e^{i\theta}$ for $z \neq 0$, then we have the equation

$$(*) \quad \operatorname{Re} (\zeta^{2k} z^{2k}) + P(z) = |z|^{2k} \operatorname{Re} Q_\zeta(e^{i\theta}).$$

Now the sectors in the complex line $w = \zeta z$ divide the unit circle $|z| = 1$ into a collection of arcs, and equation $(*)$ says that these arcs are the components of the intersection with that circle of the set

$$E_\zeta = \{z \in \mathbf{C}: \operatorname{Re} Q_\zeta(z) < 0\}.$$

Now instead of studying the sectors we study E_ζ . The intersection of E_ζ with the closed unit disc $\overline{\mathbf{B}}$ has a relatively simple structure because P is subharmonic. We now describe this structure.

A straightforward computation reveals that the condition that P be subharmonic is equivalent to

$$\operatorname{Re} [4k^2 Q_\zeta(z) - zQ'_\zeta(z) - z^2 Q''_\zeta(z)] \geq 0 \text{ for } |z| = 1.$$

Here we use the prime notation to indicate the derivative with respect to z . Because the function on the left is harmonic on the plane and nonzero at 0 (since $a_0 = c_k > 0$), this condition in turn is equivalent to

$$\operatorname{Re} [4k^2 Q_\zeta(z) - zQ'_\zeta(z) - z^2 Q''_\zeta(z)] > 0 \text{ for } |z| < 1.$$

It turns out that this condition implies the following: Fix ζ and assume that $E_\zeta \cap \mathbf{B}$ is nonempty. Then there exists m with $1 \leq m \leq 2k$ so that we have the disjoint unions

$$E_\zeta \cap \mathbf{B} = \bigcup_{j=1}^m W_j,$$

$$E_\zeta \cap \partial\mathbf{B} = \bigcup_{j=1}^m A_j,$$

and

$$\{z \in \mathbf{C}: \operatorname{Re} Q_\zeta(z) = 0\} \cap \overline{\mathbf{B}} = \bigcup_{j=1}^m B_j.$$

Here for each j we have that W_j is open in \mathbf{C} , A_j is an open arc, B_j is a smooth curve, and $\partial W_j = A_j \cup B_j$. Further, we have the following fact: For each j there is a unique point η_j on B_j closest to the origin,

and $\eta_j/|\eta_j| \in A_j$. Clearly, by our geometric interpretation of equation (*) above, there is a one-to-one correspondence between $\{A_j\}_{j=1}^m$ and the sectors, and we denote by S_j the sector corresponding to A_j . Then $\eta_j \in S_j$. We call η_j the distinguished point of S_j . Because any two sectors must be separated by an angular opening of size at least $\pi/(2k)$, the distinguished points are separated from each other by an angle of at least $\pi/(2k)$. This property is crucial for constructing the Riemann surface from the sectors. We remark that we may arrange (by a change of variables) that when $\zeta = 0$ the number of sectors is $2k$.

Now it is not hard to see that the distinguished points η_j are the solutions of the system of equations

$$\operatorname{Re} Q_\zeta(\eta) = \operatorname{Im} [\eta Q'_\zeta(\eta)] = 0, \quad \eta \in \mathbf{B}.$$

If we think of this as a system of equations in ζ and η , it defines a Riemann surface with projection $\pi(\zeta, \eta) = \zeta$. Here we use the fact that the distinguished points are separated, so each η_j can be viewed as a local real-analytic diffeomorphism.

Laszlo also showed how to extend the Riemann surface by considering the limit as ζ tends to ∞ . This concludes the construction of the Riemann surface. \square

Construction of the locally linear function g : Now we turn to the construction of a holomorphic function g that is linear on each sector in the cone C . This depends on locating certain zeros of Q_ζ , and the details were carried out by Laszlo in case P has the form described in the theorem. Carrying out the details requires, among other things, careful analysis of certain trigonometric polynomials. The special form of P makes this analysis possible (but still somewhat tedious).

As before, we work in the complex line $w = \zeta z$. The idea is that a given sector S_j has a distinguished point η_j , and we associate to η_j a zero α_j of Q_ζ . Recall that η_j belongs to

$$\{z \in \mathbf{C}: \operatorname{Re} Q_\zeta(z) = 0\},$$

so we are requiring that α_j belong to the intersection of that level set with the “conjugate” level set

$$\{z \in \mathbf{C}: \operatorname{Im} Q_\zeta(z) = 0\}.$$

The desired regularity of the peak function at 0 will hold if the argument of z and α_j are close enough when z belongs to S_j . (Shortly we

will explain this point in more detail.) Laszlo constructs a correspondence between the distinguished point η_j for each sector and a zero α_j in a way that gives a real-analytic diffeomorphism as the complex line varies, with the additional property that

$$|\arg \alpha_j - \arg \eta_j| < \frac{\pi}{4k}.$$

Now we conclude the proof of the theorem with a focus on how the regularity of the peak function is proved. Pick $(z, w) \in C$, and for simplicity assume that $z \neq 0$. Write $w = \zeta z$. The intersection with C of the corresponding complex line is a disjoint union of sectors S_1, \dots, S_m . Thus z belongs to S_j for some j . If (in the preceding notation) η_j is the distinguished point of S_j and α_j is the corresponding zero of Q_ζ , define

$$g(z, w) = \frac{z}{\alpha_j}.$$

The function g is holomorphic on C , is linear on each sector, and satisfies a Hölder condition of order 1 near 0. Further, because we always have

$$|\arg \alpha_j - \arg \eta_j| < \frac{\pi}{4k},$$

and because each sector has an angular opening of size at most $\pi/(2k)$, we have

$$|\arg g(z, w)| < \frac{3\pi}{4k}.$$

Now define

$$h(z, w) = \prod_{j=1}^{2k} g(z, e^{i\pi j/k} w),$$

and define a function H holomorphic on D' by

$$H(z, w^{2k}) = h(z, w)$$

and $H(0, 0) = 0$. Then H is of class $C^{1/(2k)}$ near 0. Now an examination of the equation defining C (in which w appears to the power $2k$) and the construction of g shows that if $w_0^{2k} = w$ then

$$H(z, w) = [g(z, w_0)]^{2k}.$$

Hence

$$|\arg H(z, w)| < \frac{3\pi}{2}.$$

Now the argument of H is continuous, and we conclude that H has a cube root that is a strong support function and belongs to $A^{1/(6k)}$ near 0. \square

5. Alternative constructions of peak functions

In this section we present two constructions of peak functions that are alternatives to the sector method presented in the preceding section. Both use an adaptation of the Bishop 1/4–3/4 method (see Theorem 2.7).

In [26], Fornæss and Sibony constructed peak functions on pseudoconvex domains of finite type in \mathbf{C}^2 . A key element of their proof is an existence theorem for entire functions dominated in terms of subharmonic functions, as follows.

5.1 Theorem. *Let ϕ be a subharmonic function on \mathbf{C} that is not harmonic. Assume that there exist constants m and C so that*

$$\phi(z + z') - \phi(z) \leq C \text{ when } |z'| \leq (1 + |z|)^{-m}.$$

Then for each sufficiently large λ and M there exists an entire function f so that $f(0) = 1$ and

$$|f(z)| \leq M \frac{\exp(\lambda \phi(z))}{1 + |z|} \text{ for } z \in \mathbf{C}.$$

The proof depends on Hörmander's theory of solving the $\bar{\partial}$ -equation in L^2 spaces with weights. We omit the details.

Now we can explain the approach of [26] to constructing peak functions at points of finite type.

5.2 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^2 and $p \in \partial D$ a point of finite type. Then p is a peak point for $A(D)$.*

Sketch of proof: A careful analysis of the defining function in special coordinates shows that the domain can be “bumped out” away from p . (We will explain such bumping methods in the next section.) The result is that it suffices to construct a peak function at $p = 0$ relative to the domain U defined by $\operatorname{Re} w + \phi(z) < 0$. Here ϕ is a subharmonic function that is homogeneous of degree $2k$ (the type of D at p) and of class C^∞ away from 0. Now apply Theorem 5.1 to ϕ with $m = 2k$ to

get an entire function f for some choice of λ and M . If we put, for j a positive integer,

$$f_j(z, w) = \exp(jw)f(j^{1/(2k)}\lambda^{-1/(2k)}z),$$

then $f_j(0, 0) = 1$, and for $(z, w) \in \overline{U}$ we have

$$|f_j(z, w)| \leq M/(1 + j^{1/(2k)}\lambda^{-1/(2k)}|z|).$$

Hence for each $\delta > 0$ and $\epsilon > 0$ there exists J so that if $j \geq J$ and $|z| \geq \delta$ then $|f_j(z, w)| < \epsilon$ for all w . An adaptation of the Bishop 1/4–3/4 method gives a peak function for p in $A(U)$. \square

Another approach to constructing peak functions on domains of finite type in \mathbf{C}^2 is due to Fornæss and McNeal in [24]. Their method is especially promising because it produces local peak functions on domains satisfying certain general conditions related to the Bergman kernel and the $\bar{\partial}$ -Neumann operator, both of which have been objects of intense study. A careful statement of these conditions would require considerable background material, so we only sketch the main ideas.

First we recall the definition of the Bergman kernel. Let D be a bounded domain in \mathbf{C}^n , and consider the space $\mathcal{H}^2(D)$ of holomorphic functions belonging to $L^2(D)$, equipped with the standard inner product. For fixed $q \in D$ the bounded linear functional sending $f \in \mathcal{H}^2(D)$ to $f(q)$ can be represented as the inner product of f with a function $k_q \in \mathcal{H}^2(D)$. The Bergman kernel for D is the function K on $D \times D$ defined by $K(z, q) = k_q(z)$. It is a fact that always $K(z, z) > 0$ for $z \in D$.

Here is how the properties of the Bergman kernel are used in [24] to construct approximate peak functions.

5.3 Lemma. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^2 of finite type and $p \in \partial D$. Denote by K the Bergman kernel for D . There exists a constant C so that for every neighborhood V of p the following holds: For q sufficiently close to p on the inward normal to ∂D at p , define f_q on D by*

$$f_q(z) = K(z, q)/K(q, q).$$

Then f_q is holomorphic on D , $f_q(q) = 1$, $|f_q| \leq C$ on D , and $|f_q| < 1/2$ on $D \setminus V$.

Idea of proof: Only the last two properties require verification. For the last of these the key properties of the Bergman kernel are the following. First, $K(z, z)$ tends to infinity as z approaches ∂D ; this holds

(with good control on the rate) for all smooth bounded pseudoconvex domains by the work of Pflug in [54]. This property gives a good lower bound for the denominator of f_q in terms of the distance to the boundary. Second, the Bergman kernel decays away from the boundary diagonal; this depends (via work of McNeal in [47]) on the existence of a subelliptic estimate for the $\bar{\partial}$ -Neumann problem, as proved by Catlin for pseudoconvex domains of finite type in \mathbf{C}^n . Combining these two properties shows that we can make $|f_q|$ small on $D \setminus V$ by taking q close to p . The remaining claim in the lemma is the uniform boundedness on D of the family $\{f_q\}$. This requires delicate estimates of the kernel on (biholomorphic images of) polydiscs adapted to the boundary geometry; such estimates are known to hold on domains of finite type in \mathbf{C}^2 and on certain other domains (see Section 6 below). \square

With this lemma in hand, Fornæss and McNeal adapt the Bishop 1/4–3/4 method to prove the existence of peak functions in $A(D)$. In fact, they also show how a strengthened form of the lemma gives greater regularity:

5.4 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^2 of finite type. Then there exists a constant $\alpha > 0$ so that every boundary point of D is a peak point for $A^\alpha(D)$.*

Note that the regularity exponent α is independent of the boundary point. This exponent can be expressed in terms of the maximum type of D and a constant (with a similar role to the constant C in the lemma) that arises in the proof. Their proof requires Hölder estimates for the $\bar{\partial}$ -equation to pass from local to global peak functions. As we explain in Section 6 below, Cho later showed how to avoid this dependence on estimates for the $\bar{\partial}$ -equation.

We remark that the relevance of the Bergman kernel to the existence of peak functions can be seen in the following way. Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n with Bergman kernel K . Assume that the kernel can be extended to a function, still denoted by K , on $(\bar{D} \times \bar{D}) \setminus \{(z, z) : z \in \partial D\}$ so that, for fixed $p \in \partial D$, the function $h(z) = K(z, p)$ is holomorphic on D and continuous on $\bar{D} \setminus \{p\}$. (This extension is known to hold on all pseudoconvex domains of finite type in \mathbf{C}^n , where h is known even to be smooth on $\bar{D} \setminus \{p\}$. For the case of strictly pseudoconvex domains see the seminal work of Kerzman in [37]. See also the work of Bell in [4], Boas in [7], and Chen in [11].) Then the following holds.

5.5 Proposition. *With the preceding notation, if $h(z)$ tends to infinity as z approaches p through points of D , then p is a local peak point for $A(D)$.*

Proof: The function $1/h$ can be extended to a function in $A(D \cap V)$ (for some neighborhood V of p) whose only zero in the closure of $D \cap V$ is at p . We will see in the last section (Theorem 7.2) that it follows that p is a local peak point for $A(D)$. \square

We remark that the condition on the kernel in the proposition should not be confused with the statement that $K(z, z)$ tends to infinity as z approaches the boundary from the interior. As we noted above, that statement is valid on all smooth bounded pseudoconvex domains. The condition on the kernel in the proposition is known to hold at strictly pseudoconvex boundary points and on the relatively few pseudoconvex domains of finite type for which the Bergman kernel can be computed explicitly.

6. Domains of finite type in \mathbf{C}^n

The constructions of peak functions described in Sections 4 and 5 can be extended to certain domains in \mathbf{C}^n . A recurring theme is the usefulness of bumping outward the domain near the boundary point in question.

In [2], Bedford and Fornæss show that their results in \mathbf{C}^2 imply the following. Suppose that D is a smooth bounded pseudoconvex domain in \mathbf{C}^n of finite type at p and that the Levi form at p (considered as a Hermitian form on the complex tangent space) has corank 1 (that is, it has $n - 2$ positive eigenvalues). Then p is a peak point for $A^\alpha(D)$ for some $\alpha > 0$. Of course, the rank condition is automatically satisfied in \mathbf{C}^2 .

In [24], Fornæss and McNeal show that their method involving the Bergman kernel also applies to the class of decoupled pseudoconvex domains of finite type in \mathbf{C}^n , defined as follows. Let D be a smooth bounded pseudoconvex domain of finite type at p . We say that D is decoupled near p if there exist holomorphic coordinates $z = (z_1, \dots, z_{n-1}, w)$ near p in which $p = 0$ and in which D is defined near p by

$$\operatorname{Re} w + \sum_{j=1}^{n-1} \phi_j(z_j) < 0.$$

Here each ϕ_j is a smooth subharmonic, but not harmonic, function on \mathbf{C} . Fornæss and McNeal show that if D is a decoupled pseudoconvex domain of finite type in \mathbf{C}^n then there exists a constant $\alpha > 0$ so that every boundary point of D is a peak point for $A^\alpha(D)$. Their method of constructing local peak functions is exactly the same as for domains of finite type in \mathbf{C}^2 —the required conditions on the Bergman kernel and the $\bar{\partial}$ -Neumann operator are known to be satisfied for decoupled domains.

The method of Fornæss and McNeal was extended by Cho in [12] and [13]. (See also Kolář's work in [43].) Cho uses bumping families he had constructed in earlier work to avoid the requirement of uniform estimates for the $\bar{\partial}$ -equation. (Cho's bumping fixes the boundary on a neighborhood of the given point.) His method produces a peak function at those boundary points of domains of finite type in \mathbf{C}^n near which the domain is convex. Here he uses work of McNeal estimating the Bergman kernel on convex domains. The peak function belongs to A^α (for some $\alpha > 0$) and extends holomorphically across the boundary away from arbitrarily small neighborhoods of the peak point. Cho's method also applies to the classes of pseudoconvex domains of finite type we have already mentioned: domains for which the Levi form has corank at most 1, and decoupled domains.

As we noted in Section 3, Diederich and Fornæss in [18] gave an explicit formula for a local strong support function in A^ω at each boundary point of a convex domain of finite type. Their result covers the locally convex case considered by Cho.

Now we turn to results that depend on bumping outward the domain at a given boundary point. This is a different kind of bumping than Cho's: Here a bumping of D near $p \in \partial D$ is a domain containing $\bar{D} \setminus \{p\}$ near p . In [2], Bedford and Fornæss show how to extend the sector method to the domain D in \mathbf{C}^n defined by

$$\operatorname{Re} w + P(z') < 0,$$

where P is a plurisubharmonic polynomial on \mathbf{C}^{n-1} that is homogeneous of degree $2k > 0$ and satisfies the following condition: The function P_ϵ on \mathbf{C}^{n-1} defined by

$$P_\epsilon(z') = P(z') - \epsilon|z_1|^{2k} - \cdots - \epsilon|z_{n-1}|^{2k}$$

is plurisubharmonic for some $\epsilon > 0$. (Recall that an upper semicontinuous function is plurisubharmonic if it is subharmonic along each

complex line. Such a function u is strictly plurisubharmonic if for each point there exists $\epsilon > 0$ so that the function $u(z) - \epsilon||z||^2$ is plurisubharmonic near that point.) Thus the domain defined by

$$\operatorname{Re} w + P_\epsilon(z') < 0$$

represents a bumping of D . Now we explain why the condition on P and the existence of a bumping are important for the construction of peak functions by Bedford and Fornæss.

As in the construction in \mathbf{C}^2 , the first step in \mathbf{C}^n is to intersect the cone C defined by

$$\operatorname{Re} (w^{2k}) + P(z') < 0$$

with complex lines through 0 and to study the resulting sectors. By assumption, P is strictly subharmonic away from the origin on each such complex line. This implies that the sectors vary smoothly with the complex line, so they may be considered as points of a Riemann domain S over complex projective space of dimension $n - 1$. To obtain the desired function g linear on each sector, one needs that S is Stein. In [2], Bedford and Fornæss show that the condition on P implies that S is locally pseudoconvex. Then S is Stein by Fujita's solution of the Levi problem for Riemann domains over projective space.

So far the existence of a bumping domain has not been used directly. But the construction of peak functions—in particular the holomorphic extension across the boundary away from the origin—requires that g be defined on a conical neighborhood of C . In \mathbf{C}^2 , this is accomplished in [2] by examining the projection of the boundary of S to complex projective space. In \mathbf{C}^n , under the assumptions on P one constructs a larger Riemann domain S_ϵ by applying the sector construction to the cone C_ϵ defined by

$$\operatorname{Re} (w^{2k}) + P_\epsilon(z') < 0.$$

Note that S is a relatively compact subset of S_ϵ and C_ϵ is a conical neighborhood of C . The desired function g on C_ϵ is then obtained from a section of a line bundle over S_ϵ .

We remark that, because of the bumping procedure, the construction of Bedford and Fornæss is not restricted to pseudoconvex domains with a defining function of the form $\operatorname{Re} w + P(z')$. To explain this, we recall from the beginning of Section 3 the function A appearing in a standard form for a defining function. The method of Bedford and Fornæss produces a peak function at every boundary point for which

the initial homogeneous term P of the expansion of A satisfies the condition in their result (namely, that the associated function P_ϵ be plurisubharmonic for some $\epsilon > 0$). Thus the defining function for the general domain has the form

$$\operatorname{Re} w + P(z') + O(|\operatorname{Im} w|^2 + \|z'\| |\operatorname{Im} w| + \|z'\|^{2k+1}).$$

We call the domain defined by $\operatorname{Re} w + P(z') < 0$ a homogeneous model for such a general domain. The point is that the higher-order terms in the defining function are controlled by means of the conical neighborhood.

In [51], Noell showed how to weaken the assumption on the leading homogeneous term P . He showed that if P is not harmonic on any complex line through the origin then 0 is a peak point for A^α (for some $\alpha > 0$) relative to the model domain, and is in fact a peak point for any smooth bounded pseudoconvex domain with such a model. The key proposition in [51] is the following bumping result.

6.1 Proposition. *Let P be a homogeneous plurisubharmonic polynomial on \mathbf{C}^{n-1} . Assume that P is not harmonic on any complex line through the origin. Then there exists a function F on $\mathbf{C}^{n-1} \setminus \{0'\}$ that is smooth, positive, and homogeneous of the same degree as P , and so that $P - F$ is strictly plurisubharmonic.*

Note that F need not be smooth at the origin.

Sketch of proof: The proof uses the original geometric setting: Let D be the homogeneous model defined by

$$\operatorname{Re} w + P(z') < 0.$$

The function F is described in terms of a family of pseudoconvex neighborhoods of $\overline{D} \setminus \{z' = 0\}$ that are strictly pseudoconvex away from $\{z' = 0\}$. These neighborhoods are constructed using a modification of the technique introduced by Diederich and Fornæss in [16]. Their technique requires the existence of a certain stratification of the boundary, and in the case of D such a stratification exists by the work of Diederich and Fornæss in [17].

The strictly pseudoconvex neighborhoods have defining functions of the form

$$\operatorname{Re} w + P(z') - \psi(z') \|z'\|^{2k-2},$$

where $2k$ is the degree of P , ψ is homogeneous of degree 2, and away from the origin ψ is smooth and positive. Once such a function ψ is constructed to give a domain that is strictly pseudoconvex away from $\{z' = 0\}$, one can define $F(z') = \psi(z')||z'||^{2k-2}$. To construct ψ , note that the homogeneity of P implies that the projection of each stratum to \mathbf{C}^{n-1} can be taken to be invariant under dilations. Locally ψ is defined to be a large negative multiple of the distance squared to such a projected stratum, plus $||z'||^2$. \square

The condition on P in Noell's result is equivalent to requiring that the homogeneous model

$$\operatorname{Re} w + P(z') < 0$$

be pseudoconvex and of finite type at the origin. His result does not apply to general pseudoconvex domains of finite type for the following reason: Even when the function A (from the standard form for a defining function) does not vanish on any nontrivial complex variety, the leading homogeneous term P of A may vanish on a complex line through the origin. For example, define A on \mathbf{C}^2 by $A(z_1, z_2) = |z_1|^2 + |z_2|^4$. Then $P(z_1, z_2) = |z_1|^2$, which vanishes on the complex line $z_1 = 0$. In this example assigning weight 2 to z_1 and weight 1 to z_2 will avoid this difficulty. In [34], Herbort showed how to construct a peak function on certain weighted homogeneous model domains. His method uses Hörmander's theory of solving the $\bar{\partial}$ -equation in L^2 spaces with weights, as in the construction of peak functions by Fornæss and Sibony.

To discuss the general weighted homogeneous case we use the terminology of Yu in [65] (see also [63], [64], and [66]), who proved the definitive results along these lines. Quite similar results were proved by Diederich and Herbort in [20]. A multiindex $\Lambda = (\lambda_1, \dots, \lambda_m)$ is called a multiweight if $1 \geq \lambda_1 \geq \dots \geq \lambda_m > 0$. A function h on \mathbf{C}^m is said to be Λ -homogeneous if

$$h(t^{\lambda_1} z_1, \dots, t^{\lambda_m} z_m) = t h(z_1, \dots, z_m)$$

for every $t > 0$ and every $(z_1, \dots, z_m) \in \mathbf{C}^m$. Let P be a real-valued function on \mathbf{C}^{n-1} that is smooth on $\mathbf{C}^{n-1} \setminus \{0'\}$, and define

$$D = \{(z', w) \in \mathbf{C}^n : \operatorname{Re} w + P(z') < 0\}.$$

We say that D is a Λ -homogeneous model if P is Λ -homogeneous and plurisubharmonic but not pluriharmonic (i.e., not the real part of a

holomorphic function), and we say that D is a Λ -homogeneous polynomial model if in addition P is a polynomial. If D is a Λ -homogeneous model with function P , then D is called h -extendible if there exists a Λ -homogeneous function F on \mathbf{C}^{n-1} that on $\mathbf{C}^{n-1} \setminus \{0'\}$ is of class C^1 and positive, and so that $P - F$ is plurisubharmonic. (Compare this definition with the conclusion of Proposition 6.1.) The following bumping result is proved in [65] (see also [20]).

6.2 Theorem. *Every Λ -homogeneous polynomial model of finite type is h -extendible.*

The bumping function F can be taken to be smooth away from the origin. The proof is formally similar to that of Proposition 6.1, but it is much more difficult and depends on results of Catlin in [9].

Earlier, in [63] (see also [65]), Yu had showed how to construct peak functions on h -extendible models:

6.3 Theorem. *If D is an h -extendible model, then 0 is a peak point for $A(D)$.*

The method uses Hörmander's theory of solving the $\bar{\partial}$ -equation in L^2 spaces with weights, as in the construction of peak functions by Fornæss and Sibony.

By combining Theorem 6.2 and Theorem 6.3, one gets a peak function at the origin (extending holomorphically across the boundary away from the origin) for every Λ -homogeneous polynomial model of finite type.

To explain how these results apply to domains more general than models, we need more definitions. Fix a boundary point p of a smooth bounded pseudoconvex domain in \mathbf{C}^n . For $1 \leq q \leq n$ we denote by $\Delta_q(p)$ the maximum order of contact of q -dimensional complex analytic varieties with the boundary at p , the D'Angelo q -type. Now assume that p is a point of finite type (so $\Delta_1(p)$ is finite). In [9], Catlin defined the multitype at p . The actual definition is somewhat complicated, but briefly the multitype is an n -tuple $(m_1(p), \dots, m_n(p))$ of rational numbers determined by measuring orders of vanishing of a defining function in the coordinate direction z_j with the weight $m_j(p)$ attached. Always $m_1(p) = 1$, $m_2(p) = \Delta_{n-1}(p)$, and $m_1(p) \leq m_2(p) \leq \dots \leq m_n(p)$. Further, if the Levi form has rank q at p then $m_j(p) = 2$ when $2 \leq j \leq q + 1$ and $m_j(p) > 2$ when $j > q + 1$. Catlin proved that if $1 \leq q \leq n$ then

$$m_{n+1-q}(p) \leq \Delta_q(p).$$

Here is the main result of [20] and [65] with regard to peak functions on bounded domains.

6.4 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n and p a boundary point of finite type. If*

$$m_{n+1-q}(p) = \Delta_q(p)$$

when $1 \leq q \leq n$, then p is a peak point for $A(D)$.

In [20], in addition it is proved that there exists a local peak function in A^α for some $\alpha > 0$, so there is a global peak function in some A^α if, say, D is of finite type.

The proof of the theorem depends on relating the domain to a model domain. To explain this we use the notation of the theorem and define the multiweight $\Lambda = (1/m_2(p), \dots, 1/m_n(p))$. If r is a defining function near p , then Catlin's work gives local holomorphic coordinates (z', w) in which $p = 0$ and

$$r(z', w) = \operatorname{Re} w + P(z') + R(z', w),$$

where P is a Λ -homogeneous plurisubharmonic polynomial with no pluriharmonic terms and R is a smooth function of strictly higher order than P (in terms of the coordinates weighted according to Catlin's multitype). Then

$$\operatorname{Re} w + P(z') < 0$$

defines a Λ -homogeneous polynomial model associated to D at p . We say that D is h -extendible at p if it has an associated Λ -homogeneous polynomial model that is h -extendible. Yu proved that in this case there exists an h -extendible model D' so that $\overline{D} \setminus \{p\}$ is contained in D' near p . Now we can state the remarkable characterization of h -extendibility given in [20] and [65].

6.5 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n and p a boundary point of finite type. Then D is h -extendible at p if and only if*

$$m_{n+1-q}(p) = \Delta_q(p)$$

when $1 \leq q \leq n$.

We omit the proof. Now the proof of Theorem 6.4 is clear: Apply Theorem 6.5 to see that D is h -extendible at p , and use Yu's result to get an h -extendible model D' so that $\overline{D} \setminus \{p\}$ is contained in D'

near p . Then apply the earlier construction of peak functions on Λ -homogeneous polynomial models of finite type to get a peak function relative to D' at p . This function is also a local peak function relative to D and extends holomorphically across the boundary away from p .

We note that Theorem 6.4 implies all of the results presented in this section on the peak function problem: All of the classes of bounded domains considered in this section—domains for which the Levi form has corank at most 1, decoupled domains, and convex domains—are known to be h -extendible.

Recently, Bharali and Stensønes have proved bumping results for certain domains in \mathbf{C}^3 that are not h -extendible. See [5].

7. Miscellany

In this section we list some miscellaneous results related to the peak point problem and its generalizations.

One natural question is whether peak functions can be chosen to vary in a regular way from point to point. In [31], Graham showed that on smooth bounded strictly pseudoconvex domains one can find peak functions in A^ω that vary continuously with the boundary point. Fornæss and Krantz proved a very general result in [23]: If X is a compact metric space and \mathcal{A} is a closed subalgebra of $C(X)$ whose set of peak points is \mathcal{P} , then there is a continuous function $\Phi: \mathcal{P} \rightarrow \mathcal{A}$ such that $\Phi(x)$ is a peak function for x when $x \in \mathcal{P}$.

One important result on peak points proved by Gamelin in [29] involves an approach very different from any considered so far. Recall that a subset K of \mathbf{C}^n is said to be circled if $(\lambda_1 z_1, \dots, \lambda_n z_n) \in K$ whenever $(z_1, \dots, z_n) \in K$ and $\lambda_1, \dots, \lambda_n$ are complex numbers of modulus 1. Let K be a compact, connected, circled subset of \mathbf{C}^n . We denote by $H(K)$ the uniform closure in $C(K)$ of the functions holomorphic in a neighborhood of K . The rational convex hull of K , denoted $r(K)$, is the set of points $z \in \mathbf{C}^n$ with the property that every holomorphic polynomial that vanishes at z also has a zero on K . (For such K the set $r(K)$ may be identified with the spectrum of $H(K)$.) Here is the main result from [29].

7.1 Theorem. *Let K be a compact, connected, circled subset of \mathbf{C}^n . If $p \in K$ is not a peak point for $H(K)$, then p belongs to an analytic disc that is contained in $r(K)$.*

The proof uses methods from the theory of function algebras.

In the introduction we noted that analytic structure in the boundary is an obstruction to the existence of peak functions. In [67], Yu showed that even without such analytic structure a peak function may fail to exist: He constructed a smooth bounded pseudoconvex domain D in \mathbf{C}^3 so that ∂D contains no nontrivial analytic set and so that some boundary point is not a local peak point for $A(D)$. The domain constructed by Yu is not of finite type, but it is B-regular, which means roughly that there is no pluripotential theory on the boundary. (See Sibony's survey paper [60] for a discussion of B-regularity.) An example of a related phenomenon was given by Noell in [49]: There is a smooth bounded convex domain D in \mathbf{C}^2 so that D is strictly pseudoconvex except along a line segment and so that, for all $\alpha > 0$, each point of that segment is not a peak point for $A^\alpha(D)$. The domain is of infinite type along the line segment, and every point of the segment is a peak point for $A(D)$.

The notion of a peak point can be generalized as follows. If D is a domain in \mathbf{C}^n with smooth boundary, a compact subset K of ∂D is a peak set relative to D for a space \mathcal{A} if there exists a function $f \in \mathcal{A}$ so that $f = 1$ on K and $|f| < 1$ on $\overline{D} \setminus K$. As for peak points, we use the terms peak function and strong support function. Peak sets have been the subject of extensive study, but even in the case $n = 1$ a complete characterization of peak sets for A^α is unknown if $0 < \alpha < 1$.

We state here an open question that is a natural generalization of the peak point problem.

Question. *If D is a smooth bounded pseudoconvex domain in \mathbf{C}^n , does every boundary point of D belong to a peak set for $A(D)$?*

An affirmative answer would imply completeness in the standard invariant metrics. We remark that clearly every boundary point of a bounded convex domain belongs to a peak set for A^ω .

A result from the theory of function algebras implies that if D is a bounded domain then every peak set for $A(D)$ contains a peak point for $A(D)$. (See Corollary 2.4.6 of [8].) In the example of Noell mentioned above, the line segment is a peak set for A^ω . This illustrates the fact that the result from function algebra theory does not extend to A^α for $\alpha > 0$.

Sets more general than peak sets have also been studied. If D is a domain in \mathbf{C}^n with smooth boundary, a compact subset K of ∂D is a zero set relative to D for a space \mathcal{A} if there exists a function $f \in \mathcal{A}$ so that $f = 0$ on K and $f \neq 0$ on $\overline{D} \setminus K$. We call f a support function. Of

course, every peak set is a zero set, and each strong support function is a support function.

The original paper [2] of Bedford and Fornæss constructed a support function in A^∞ at each boundary point of pseudoconvex domains of finite type in \mathbf{C}^2 . The support function they construct vanishes to infinite order at the point in question. As noted in [2], the argument of Kohn and Nirenberg in [41] shows that for the Kohn-Nirenberg domain every support function for the origin must vanish to infinite order there.

At the other extreme from the case of infinite order of vanishing, we have the case when, for a point p in the boundary of a domain D , there is a local support function $f \in A^\omega(D)$ with nonzero gradient at p . Then it follows that there is a complex submanifold X of complex codimension 1 in a neighborhood V of p so that $X \cap \overline{D} \cap V = \{p\}$. We call such a set X a local support manifold for D at p . We remark that, by the holomorphic Hopf lemma (2.6), if p is a local peak point for A^ω then there is a local support manifold at p . We refer to Théorème 1 (and its proof) in the paper [33] of Hakim and Sibony for a very useful generalization of this fact that covers the case when p belongs to a peak set for $A^\infty(D)$, with D of finite type at p .

The existence of a local support manifold is closely related to the strict type conditions considered in Section 3. In fact, Range's strict type condition (the weakest of the three conditions considered) at a point clearly implies the existence of a local support manifold there. For the converse, in [6] Bloom observed that if there is a local support manifold for p , and if the boundary is real-analytic near p , then Range's strict type condition holds at p .

The example of Bloom that we considered in Proposition 3.6 has a local support manifold at the origin (because the domain is of strict type in the sense of Range at the origin). This example shows that the existence of a smooth peak function does not follow from the existence of a local support manifold. It is true, however, that if there is a local support manifold at a boundary point p of a smooth bounded domain D then p is a local peak point for $A(D)$. (If also D is pseudoconvex then p is a global peak point for $A(D)$.) This was proved by Hakim and Sibony in [33] and by Range in [56]. One definitive result along these lines was proved by Verdera in [61]:

7.2 Theorem. *Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n . If $K \subset \partial D$ is a zero set for $A(D)$ then K is a peak set for $A(D)$.*

The elegant proof uses results such as the identification of the spec-

trum of $A(D)$ and the Arens-Royden Theorem. In the case of strictly pseudoconvex domains this theorem was proved by Chaumat and Chollet in [10] and by Weinstock in [62]. Here is a proof (from [10]) for the case when D is simply connected: Let D be a simply connected domain and $K \subset \partial D$ a zero set for $A(D)$, with support function f . We may assume that $|f| < 1$ on \overline{D} . Then there exists a function ϕ holomorphic on D so that $\exp \phi = f$, and we have $\operatorname{Re} \phi < 0$ on D . It is easy to see that $-1/\phi$ extends to a strong support function for K in $A(D)$. Hence K is a peak set for $A(D)$. We remark that as a special case we have that if D is any smooth bounded domain with $p \in \partial D$, and if $\{p\}$ is a local zero set for $A(D)$, then p is a local peak point for $A(D)$. This local result is all we needed in the proof of Proposition 5.5.

We also mention the notion of a plurisubharmonic peak function for a domain D at a point $p \in \partial D$, namely, a function $\psi \in C(\overline{D})$ that is plurisubharmonic on D and satisfies $\psi(p) = 0$ and $\psi < 0$ on $\overline{D} \setminus \{p\}$. Sibony showed (see Theorem 2.3 of [60]) that a smooth bounded pseudoconvex domain in \mathbf{C}^n is B-regular if and only if there is a plurisubharmonic peak function at each boundary point. Clearly every peak point for $A(D)$ has a plurisubharmonic peak function, but in light of Yu's example above the converse is false. The construction of peak functions by Fornæss and Sibony in [26] does suggest a connection between plurisubharmonic peak functions and (holomorphic) peak functions in certain settings: One of the main results of [26] is the existence, on each pseudoconvex domain of finite type in \mathbf{C}^2 , of a family of plurisubharmonic peak functions (one for each boundary point) satisfying optimal estimates and with uniform control on regularity. (They prove a similar result on convex domains of finite type in \mathbf{C}^n .) We remark that for some applications the existence of plurisubharmonic peak functions is sufficient. In [26], for example, Fornæss and Sibony show how to deduce optimal subelliptic estimates for the $\overline{\partial}$ -Neumann problem from the family described above. See also Herbort's work in [35] connecting plurisubharmonic peak functions to estimates for the Bergman kernel and metric.

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